Uniformly exponentially stable approximations for Timoshenko beams

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Abstract

In this note, Timoshenko beams with interior damping and boundary damping are studied from the viewpoints of control theory and numerical approximation. Especially, the uniform exponential stabilities of the beams are studied. The meaning of uniform exponential stability in this paper is two-fold: The first one is in the classical sense and also is concisely called exponential stability by many authors; The second one is that the semi-discretization systems, which are derived from an exponentially stable continuous beam by some semi-discretization schemes, are uniformly exponentially stable with respect to the discretized parameter. To investigate uniform exponential stability of continuous and discrete systems, five completely different methods, which are stability theory of port-Hamiltonian system, direct method of Lyapunov functional, perturbation theory of C_0-semigroup, spectral analysis of unbounded operator and frequency standard of exponential stability for contractive semigroup, are involved. Especially, a new method, which is based on the frequency domain characteristics of uniform exponential stability of C_0-semigroup of contractions, is established to verify the uniform exponential stability of semi-discretization systems derived from coupled system. The effectiveness of the numerical approximating algorithms is verified by numerical simulations.

Keywords: Timoshenko beam, exponential stability, semi-discretization, finite difference, C_0-semigroup.

1. Introduction

In this paper, we study classical Timoshenko beams with interior damping and boundary damping, which was extensively investigated in the past decades due to its widely applications in engineering [1–12], both from PDEs and numerical approximation point of view. Adding the effect of shear as well as the effect of rotation to the Euler-Bernoulli beam, Timoshenko developed a beam model which is depicted by a family of equations as follows:

$$\begin{cases} \rho w_{tt}(s,t) - K(w_s(s,t) - \phi(s,t))_s = 0, \ s \in (0,1), \ t \geq 0, \\ I_\rho \phi_{tt}(s,t) - EI\phi_{ss}(s,t) - K(w_s(s,t) - \phi(s,t)) = 0, \end{cases} \tag{1.1}$$

where t is the time variable and s is the space coordinate along the beam in its equilibrium position. $w(s,t)$ is the transverse displacement of the beam and $\phi(s,t)$ is the rotation angle of a filament of the beam. The coefficients ρ, I_ρ, E, I and K are the mass per unit length, the rotary moment of inertia of a cross section, the Youngs modulus of elasticity, the moment of inertia of a cross section and the shear modulus, respectively.

*Corresponding author. This paper was carried out with the support of the National Natural Science Foundation of China under grant No. 11871117; and Key Laboratory of Engineering Modeling and Statistical Computation of Hainan Province; and the Specific Research Fund of the Innovation Platform for Academicians of Hainan Province; and Scientific Research Initiation Fund of Hainan University (RZ2200001240).

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Many authors tried to stabilize the Timoshenko beam by designing boundary damping or interior damping in the past decades. Kim and Renardy [1] imposed the boundary conditions

$$w(0,t) = 0,\ \phi(0,t) = 0 \tag{1.2}$$

at $s = 0$ and the boundary damping

$$w_s(1,t) - \phi(1,t) = -\alpha_1 \rho w_t(1,t), \quad \phi_s(1,t) = -\alpha_2 I_\rho \phi_t(1,t). \tag{1.3}$$

at $s = 1$ for $\alpha_1,\ \alpha_2 > 0$. These boundary damping corresponds to a feedback mechanism which monitors $w_t(s,t)$ and $\phi_t(s,t)$ at $s = 1$ and transforms them into the lateral force and moment applied at $s = 1$, respectively. They showed that the energy decays exponentially by the semigroup theory and the method of Lyapunov functional. Xu and Feng [2] gave detailed spectral analysis and Riesz basis property of the generalized eigenvector system of the beam (1.1)-(1.3). Villegas, Zwart, Gorrec, and Jacob et al. [3, 6] obtained the exponential stability of (1.1)-(1.3) by the stability theory of port-Hamiltonian system on infinite-dimensional space. Raposo et al. [7] added frictional dissipative terms $w_t(s,t)$ and $\phi_t(s,t)$ into (1.1) to get

$$\begin{cases} \rho w_{tt}(s,t) - K(w_s(s,t) - \phi(s,t))_s + w_t(s,t) = 0,\ s \in (0,1),\ t \geq 0, \\ I_\rho \phi_{tt}(s,t) - EI\phi_{ss}(s,t) - K(w_s(s,t) - \phi(s,t)) + \phi_t(s,t) = 0, \\ w(0,t) = w_s(1,t) - \phi(1,t) = \phi_s(1,t) = \phi(0,t) = 0, \end{cases} \tag{1.4}$$

which is a Timoshenko beam with interior damping, and obtained the exponential stability of (1.4) by the frequency domain characteristics of exponential stability of C_0-semigroup. Yan et al. [8] studied the stabilization problem of Timoshenko beam (1.1) in the presence of linear dissipative boundary feedback controls and presented various necessary and sufficient conditions for the system to be asymptotically stable. The equivalence between the exponential stability and the asymptotic stability for the closed-loop system was finally given by the same method of [7].

Recently, Rivera and Naso [9] imposed boundary dissipation only on one side of the bending moment, i.e., $EI\phi_s(0,t) = -\alpha_2 \phi_t(0,t)$, and other three Dirichlet boundary conditions to (1.1). They proved the exponential stability of the resulting Timoshenko beam, provided the wave speeds of the system are equal. Almeida Júnior, Ramos and Freitas [10] found some new facts related to the classical Timoshenko system. More precisely, they proved the damped shear beam model, which corresponds to a part of the classical Timoshenko beam model, possessed an energy exponential decay under boundary conditions of Dirichlet-Neumann.

In this note we shall study the beam (1.1)-(1.3) and (1.4) in both continuous and discrete levels. In the continuous case, the exponential stability of (1.1)-(1.3) is analyzed by the stability theory of infinite-dimensional port-Hamiltonian systems [3, 6]. However, the exponential stability of (1.4) is investigated combining the spectral analysis (see e.g. [11, 12]) and perturbation theory of exponential stability C_0-semigroup [13]. It should be pointed that the results of exponential stabilities of (1.1)-(1.3) and (1.4) are not new, we include them to compare with the discrete results.

Whereas in the discrete case, in which main contributions of this paper are presented, a spacial semi-discretization process is firstly carried out for (1.1)-(1.3) or (1.4). A family of discrete systems are obtained and a natural problem is posed: whether or not they have an exponential decay rate independent of the step size. This is an important problem in studying of dynamical system, see for instance [14, 15] and the references therein. To answer this question, we should overcome two difficulties. The first one is how to choose an appropriate algorithm from many discretization methods. The second one is how to verify that the discrete systems possess uniform exponential stability for a given discrete scheme. Fortunately, some related researches [16–22] have been carried out for wave equation since 1990s and a series of successful results have been derived. For example, a new kind of central difference [20–22] was introduced to bypass the first obstacle and the multiplies method was widely and conveniently used to verify that the discrete systems possess uniform exponential decay.

However, a relatively small amount of work has been made for the Timoshenko beam system, see [23–25] for uniform controllability, [26] for uniform observability, and [27] for uniform exponential stability of Euler-Bernoulli beam.

We shall fill the gap and investigate the uniform exponential stability of Timoshenko beams (1.1)-(1.3) and (1.4). For this purpose, the Timoshenko beam (1.1)-(1.3) and (1.4) are reduced to a family of first order hyperbolic PDEs and they can be rewritten as the form of port-Hamiltonian system on a Hilbert space [6, Section 7.1]. A new discretized scheme, which is called the average central-difference, is introduced to the order reduction systems motivated by [20–22, 28]. However, to the best of our knowledge, suitable discrete multipliers are hard to find for the discrete systems because the Timoshenko beam (1.1) is not a simple combination of two wave equations but contains coupled terms. Therefore, applying the results of the wave in [20–22] to Timoshenko beam is nontrivial and we should deal with it with additional efforts. Finally, perturbation theory of semigroup and multiplier method are combined to overcome the second challenge for the semi-discretization systems of (1.1)-(1.3). But a new method, which is based on the frequency domain characteristics of uniform exponential stability of C_0-semigroup of contractions (see e.g. [29, P.162] and [30]), is established to verify the uniform exponential stability of semi-discretization systems of (1.4).

The structure of this paper is organized as follows. In the Section 2, the Timoshenko beam (1.1)-(1.3) is studied in both continuous and discrete case. In the Section 3, the Timoshenko beam (1.4) is studied similarly by different methods. In section 4, we give some concluding remarks.

2. Related results of the beam (1.1)-(1.3)

We discuss uniform exponential stability of Timoshenko beam (1.1) associated with the boundary conditions (1.2)-(1.3) in this section. Main results related to the continuous case and its discrete case are given in Subsection 2.1 and Subsection 2.2, respectively.

2.1. Exponential stability of continuous beam (1.1)-(1.3)

To investigate uniform exponential stability conveniently in the next subsection, we take the transformation

$$y_1(s,t) = w_s(s,t) - \phi(s,t),\ y_2(s,t) = \rho w_t(s,t),\ y_3(s,t) = \phi_s(s,t),\ y_4(s,t) = I_\rho \phi_t(s,t),$$

and obtain equivalent system of (1.1)-(1.3)

$$\begin{cases} \dot{y}_1(s,t) = \beta_2 y_2'(s,t) - \beta_4 y_4(s,t), & \dot{y}_2(s,t) = \beta_1 y_1'(s,t), \\ \dot{y}_3(s,t) = \beta_4 y_4'(s,t), & \dot{y}_4(s,t) = \beta_3 y_3'(s,t) + \beta_1 y_1(s,t), \\ y_1(1,t) = -\alpha_1 y_2(1,t), & y_2(0,t) = 0, \\ y_3(1,t) = -\alpha_2 y_4(1,t), & y_4(0,t) = 0, \end{cases} \tag{2.1}$$

in which $\beta_1 = K$, $\beta_2 = \rho^{-1}$, $\beta_3 = EI$ and $\beta_4 = I_\rho^{-1}$. Here and Hereafter, the dot and the prime represent derivatives with respect to time and spacial variables, respectively. Set $Y(s,t) = (y_1(s,t), y_2(s,t), y_3(s,t), y_4(s,t))^\top$ and the first four equations of (2.1) can be written as port-Hamiltonian system

$$\dot{Y}(s,t) = P_1[\mathcal{H}Y(s,t)]' + P_0\mathcal{H}Y(s,t), \tag{2.2}$$

in which $\mathcal{H} = \mathrm{diag}(\beta_1,\beta_2,\beta_3,\beta_4)$ and

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The state space of (2.1) or (2.2) is $\mathbb{X} := L^2([0,1];\mathbb{C}^4)$ with inner product

$$\langle Y, Z\rangle_{\mathbb{X}} = \int_0^1 Y(s)^* \mathcal{H} Z(s)\mathrm{d}s,\ \forall\, Y, Z \in \mathbb{X}. \tag{2.3}$$

The rest of this subsection is employed from [3]. To formulate the boundary conditions of (2.1) or (2.2) in a better manner, we introduce the boundary effort and boundary flow, which are defined as

$$e_\partial = \frac{1}{\sqrt{2}}\mathcal{H}[Y(1)+Y(0)] \text{ and } f_\partial = \frac{1}{\sqrt{2}}P_1\mathcal{H}[Y(1)-Y(0)], \tag{2.4}$$

respectively. Hence, we consider the operator

$$AY(s) = P_1(\mathcal{H}Y(s))' + P_0(\mathcal{H}Y(s)),\ \forall\, Y(s) \in D(A) \tag{2.5}$$

on the state space $\mathbb{X}$ and the domain

$$D(A) = \left\{ Y \in \mathbb{X} : \mathcal{H}Y \in H^1([0,1];\mathbb{C}^4),\ W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \tag{2.6}$$

in which $H^1([0,1];\mathbb{C}^4)$ is Sobolev space of one order and $W_B \in \mathbb{C}^{4\times 8}$ equals

$$W_B = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_1 & \beta_2\beta_1^{-1} & 0 & 0 & \beta_2\beta_1^{-1} & \alpha_1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_4\beta_3^{-1} & 0 & 0 & \beta_4\beta_3^{-1} & \alpha_2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, the abstract differential equation

$$\dot{Y}(s,t) = AY(s,t),\ Y(s,0) \in \mathbb{X} \tag{2.7}$$

is equivalent to (2.1). To obtain the main result of this subsection, we introduce two lemmas.

Lemma 2.1 Let I_4 be the identity operator on $\mathbb{C}^4$ and $\Sigma = \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}$, then the matrix W_B has full rank and satisfies

$$W_B \Sigma W_B^* \geq 0. \tag{2.8}$$

Proof. Let $W_B = [W_1, W_2]$ and

$$W_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_1 & \beta_2\beta_1^{-1} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_4\beta_3^{-1} \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad W_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \beta_2\beta_1^{-1} & \alpha_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta_4\beta_3^{-1} & \alpha_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have

$$\begin{aligned} &W_B\Sigma W_B^* = W_1W_2^* + W_2W_1^* \\ &= \frac{1}{2}\begin{pmatrix} 2\alpha_1\beta_2\beta_1^{-1} & -\beta_2\beta_1^{-1} & 0 & 0 \\ \beta_2\beta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 2\alpha_2\beta_4\beta_3^{-1} & -\beta_4\beta_3^{-1} \\ 0 & 0 & \beta_4\beta_3^{-1} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2\alpha_1\beta_2\beta_1^{-1} & \beta_2\beta_1^{-1} & 0 & 0 \\ -\beta_2\beta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 2\alpha_2\beta_4\beta_3^{-1} & \beta_4\beta_3^{-1} \\ 0 & 0 & -\beta_4\beta_3^{-1} & 0 \end{pmatrix} \\ &= \mathrm{diag}(2\alpha_1\beta_2\beta_1^{-1}, 0, 2\alpha_2\beta_4\beta_3^{-1}, 0) \geq 0, \end{aligned}$$

which implies (2.8) holds. $\square$

The following lemma comes from [6, Lemma 7.2.1].

Lemma 2.2 For any $Y \in D(A)$, the following results hold

$$\mathrm{Re}\,\langle AY, Y\rangle_{\mathbb{X}} = -\alpha_1\beta_1\beta_2|y_2(1,t)|^2 - \alpha_2\beta_3\beta_4|y_4(1,t)|^2. \tag{2.9}$$

Now we can show that (2.7), and hence (2.1), is exponential stable. More precisely,

Theorem 2.1 The operator A generates a C_0-semigroup of contractions $T(t)$ and $T(t)$ is exponential stable.

Proof. Theorem 7.2.4 of [6] and Lemma 2.1 imply that the first statement is right and Theorem 9.1.3 of [6] and Lemma 2.2 mean that the second statement is right. □

2.2. Uniform exponential stability of (2.1)

In this section we discuss the uniform exponential stability of Timoshenko beam (1.1)-(1.3). For this purpose, we perform the spacial semi-discretization for (2.1). Let $N \in \mathbb{N}$ be an integer and $h = 1/(N+1)$ be step size, insert $N+2$ points and $N+1$ points, which are denoted by $s_i = ih(i = 0, 1, \cdots, N+1)$ and $v_j = (j+1/2)\,h(j = 0, 1, \cdots, N)$, respectively, in the domain [0, 1]. f_i only be the value of a function $f(s)$ at node $s_i = ih(i = 0, 1, \cdots, N+1)$. This means that $y_{l,i}(t)$ denotes $y_l(s_i, t)$ for $l = 1, 2, 3, 4$. The notations $\delta_s f_{j+\frac{1}{2}} = (f_{j+1} - f_j)/h$ and $f_{j+\frac{1}{2}} = (f_{j+1} + f_j)/2$ denote the central divided difference operator of first-order derivative at $f_s(v_j)$ and the average operator at $f(v_j)$, respectively.

Now we are in a position to give the spacial semi-discretizaiton scheme of (2.1). The independent variable s in equations (2.1) is replace by v_j, i.e.,

$$\begin{cases} \dot{y_1}(v_j,t) = \beta_2 y_2'(v_j,t) - \beta_4 y_4(v_j,t), & \dot{y_2}(v_j,t) = \beta_1 y_1'(v_j,t), \\ \dot{y_3}(v_j,t) = \beta_4 y_4'(v_j,t), & \dot{y_4}(v_j,t) = \beta_3 y_3'(v_j,t) + \beta_1 y_1(v_j,t). \end{cases}$$

Approximating the spacial derivative by difference operator and applying the average operator to the time derivative respectively, we get

$$\begin{cases} \dot{y}_{1,j+\frac{1}{2}}(t) = \beta_2\delta_s y_{2,j+\frac{1}{2}}(t) - \beta_4 y_{4,j+\frac{1}{2}}(t), \\ \dot{y}_{2,j+\frac{1}{2}}(t) = \beta_1\delta_s y_{1,j+\frac{1}{2}}(t),\ j = 0, 1, \cdots, N, \\ \dot{y}_{3,j+\frac{1}{2}}(t) = \beta_4\delta_s y_{4,j+\frac{1}{2}}(t), \\ \dot{y}_{4,j+\frac{1}{2}}(t) = \beta_3\delta_s y_{3,j+\frac{1}{2}}(t) + \beta_1 y_{1,j+\frac{1}{2}}(t), \\ y_{1,N+1}(t) = -\alpha_1 y_{2,N+1}(t),\ y_{2,0}(t) = 0, \\ y_{3,N+1}(t) = -\alpha_2 y_{4,N+1}(t),\ y_{4,0}(t) = 0, \end{cases} \tag{2.10}$$

by considering the boundary conditions in (2.1).

In the state space $\mathbb{C}^{4(N+1)}$, define

$$\mathbb{X}_h = \left\{ Z_h = \begin{pmatrix} Z_{1h} \\ Z_{2h} \\ Z_{3h} \\ Z_{4h} \end{pmatrix} \in \mathbb{C}^{4(N+1)} : Z_{lh} = \begin{pmatrix} z_{l,1} \\ z_{l,2} \\ \vdots \\ z_{l,N+1} \end{pmatrix},\ Z_{kh} = \begin{pmatrix} z_{k,0} \\ z_{k,1} \\ \vdots \\ z_{k,N} \end{pmatrix},\ \begin{array}{c} z_{l,j}, z_{k,j} \in \mathbb{C}, \\ \\ k = 1, 3, l = 2, 4, \end{array} \right\}$$

with the inner product

$$\begin{cases} \langle Z_h, \widetilde{Z}_h\rangle_h = h\sum_{l=1}^4 \sum_{j=0}^N \left(\beta_l z_{l,j+\frac{1}{2}} \widetilde{z}_{l,j+\frac{1}{2}}\right), \forall Z_h, \widetilde{Z}_h \in \mathbb{X}_h, \\ z_{2,0} = z_{4,0} = 0, z_{1,N+1} = -\alpha_1 z_{2,N+1}, z_{3,N+1} = -\alpha_2 z_{4,N+1}. \end{cases} \tag{2.11}$$

The state variable of (2.10) is

$$Y_h(t) = (Y_{1h}(t), Y_{2h}(t), Y_{3h}(t), Y_{4h}(t))^\top$$

with

$$Y_{1h}(t)=(y_{1,0}(t),\cdots,y_{1,N}(t)),\ Y_{2h}(t)=(y_{2,1}(t),\cdots,y_{2,N+1}(t)),$$
$$Y_{3h}(t)=(y_{3,0}(t),\cdots,y_{3,N}(t)),\ Y_{4h}(t)=(y_{4,1}(t),\cdots,y_{4,N+1}(t)),$$

then the energy $E_h(t)$ is

$$E_h(t)=\frac{1}{2}\langle Y_h(t),Y_h(t)\rangle_{\mathbb{X}_h}=\frac{h}{2}\sum_{l=1}^{4}\sum_{j=0}^{N}\beta_l|y_{l,j+\frac{1}{2}}(t)|^2. \tag{2.12}$$

We have a discrete counterpart of Lemma 2.2

Lemma 2.3 The solution $Y_h(t)$ to the system (2.10) satisfies the balance equations of discrete version:

$$\dot{E}_h(t)=-\alpha_1\beta_1\beta_2|y_{2,N+1}(t)|^2-\alpha_2\beta_3\beta_4|y_{4,N+1}(t)|^2. \tag{2.13}$$

Proof. Multiplying the first four differential equations of (2.10) by $h\beta_l y_{1,j+\frac{1}{2}}(t)$ $(l=1,2,3,4)$ and adding them together, we obtain

$$\begin{aligned}\dot{E}_h(Y;t)=&h\beta_1\beta_2\left[\sum_{j=0}^{N}y_{1,j+\frac{1}{2}}(t)\delta_s y_{2,j+\frac{1}{2}}(t)+\sum_{j=0}^{N}y_{2,j+\frac{1}{2}}(t)\delta_s y_{1,j+\frac{1}{2}}(t)\right]\\&+h\beta_3\beta_4\left[\sum_{j=0}^{N}y_{3,j+\frac{1}{2}}(t)\delta_s y_{4,j+\frac{1}{2}}(t)+\sum_{j=0}^{N}y_{4,j+\frac{1}{2}}(t)\delta_s y_{3,j+\frac{1}{2}}(t)\right]\end{aligned}$$

However, a simple calculation shows that

$$h\left[\sum_{j=0}^{N}y_{1,j+\frac{1}{2}}(t)\delta_s y_{2,j+\frac{1}{2}}(t)+\sum_{j=0}^{N}y_{2,j+\frac{1}{2}}(t)\delta_s y_{1,j+\frac{1}{2}}(t)\right]=y_{1,N+1}(t)y_{2,N+1}(t)-y_{1,0}(t)y_{2,0}(t),$$
$$h\left[\sum_{j=0}^{N}y_{3,j+\frac{1}{2}}(t)\delta_s y_{4,j+\frac{1}{2}}(t)+\sum_{j=0}^{N}y_{4,j+\frac{1}{2}}(t)\delta_s y_{3,j+\frac{1}{2}}(t)\right]=y_{3,N+1}(t)y_{4,N+1}(t)-y_{3,0}(t)y_{4,0}(t).$$

The above two identities and boundary conditions imply that the equality (2.13) holds. □

Moreover, we additionally need the following lemma (see [20–22]) to obtain main result of this subsection.

Lemma 2.4 Let $\{u_i\}_{i=0}^{N+1}$, $\{v_i\}_{i=0}^{N+1}$ and $\{w_i\}_{i=0}^{N+1}$ be sequences consisting of real numbers, then we have

$$\begin{aligned}&h\sum_{i=0}^{N}\left[\delta_s u_{i+\frac{1}{2}}v_{i+\frac{1}{2}}w_{i+\frac{1}{2}}+u_{i+\frac{1}{2}}\delta_s v_{i+\frac{1}{2}}w_{i+\frac{1}{2}}+u_{i+\frac{1}{2}}v_{i+\frac{1}{2}}\delta_s w_{i+\frac{1}{2}}+\frac{h^2}{4}\delta_s u_{i+\frac{1}{2}}\delta_s v_{i+\frac{1}{2}}\delta_s w_{i+\frac{1}{2}}\right]\\&=u_{N+1}v_{N+1}w_{N+1}-u_0v_0w_0.\end{aligned}$$

To represent (2.10) into vectorial form, we introduce some matrices. Let A_h, B_h, C_h, and D_h be square matrices of order $N+1$:

$$A_h=\frac{1}{2}\begin{pmatrix}1&1&&\\&\ddots&\ddots&\\&&1&1\\&&&1\end{pmatrix},\quad B_h=\frac{1}{h}\begin{pmatrix}1&-1&&\\&\ddots&\ddots&\\&&1&-1\\&&&1\end{pmatrix},$$

$C_h = \mathrm{diag}(0, \cdots, 0, -\alpha_1)$, $D_h = \mathrm{diag}(0, \cdots, 0, -\alpha_2)$, respectively. Set

$$\mathcal{H}_h = \mathrm{diag}(\beta_1 I_{N+1}, \beta_2 I_{N+1}, \beta_3 I_{N+1}, \beta_4 I_{N+1}),$$

$$\Phi_h = \begin{pmatrix} A_h & 2^{-1}C_h & 0 & 0 \\ 0 & A_h^\top & 0 & 0 \\ 0 & 0 & A_h & 2^{-1}D_h \\ 0 & 0 & 0 & A_h^\top \end{pmatrix},$$

$$\Psi_h = \begin{pmatrix} 0 & \beta_2 B_h^\top & 0 & -\beta_4 A_h^\top \\ -\beta_1 B_h & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 B_h^\top \\ \beta_1 A_h & 0 & -\beta_3 B_h & 0 \end{pmatrix},$$

$$\Omega_h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta_1 h^{-1} C_h & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2^{-1}\beta_1 C_h & 0 & \beta_3 h^{-1} D_h \end{pmatrix}.$$

After these preparations, we can rewrite (2.10) as

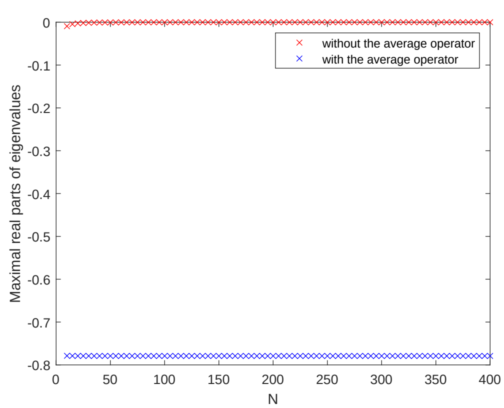
$$\dot{Y}_h(t) = \mathcal{A}_h Y_h(t), \quad Y_h(0) \in \mathbb{X}_h, \tag{2.14}$$

where $\mathcal{A}_h = \Phi_h^{-1}(\Psi_h + \Omega_h)$ because Φ_h is evidently invertible.

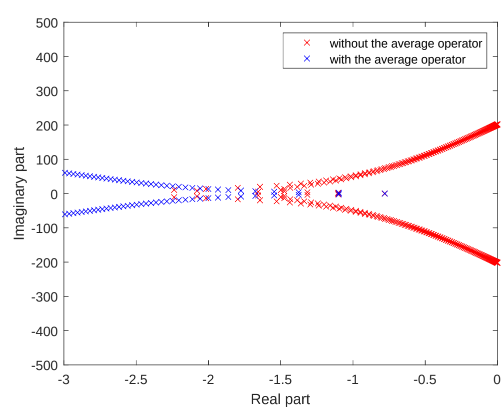
It is easy to see that Φ_h is corresponding to the average operator of time derivative of (2.10). If one replace Φ by identity operator, then the classical finite difference scheme of (2.1) is easily restored from (2.14), i.e.,

$$\dot{Y}_h(t) = (\Psi_h + \Omega_h) Y_h(t), \quad Y_h(0) \in \mathbb{X}_h. \tag{2.15}$$

Here we explain the significance of the discrete scheme (2.14). We plot two figures in Figures 1 and 2, respectively. Figure 1 depicts the maximal real parts of the eigenvalues of our discrete scheme (2.14) and the classical semi-discrete scheme (2.15) for $N = 10 : 5 : 400$. Figure 2 depicts the distributions of the eigenvalues of (2.14) and (2.15) in which $N = 100$. We see that the real parts of the eigenvalues



(a) Fig.1. Maximal real parts of eigenvalues



(b) Fig.2. Eigenvalue distributions

of (2.15) approach to zero and those of (2.14) approach to a negative number from two figures. In both figures, we take $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$, $\alpha_1 = \alpha_2 = 1$. Numerical simulation results show that (2.15) is not uniformly exponentially stable. This conclusion is consistent with earlier researches of [16]. However, Figure 1 manifests (2.14) is perhaps uniformly exponentially stable. In the remaining part of this subcection, we give strict proof about uniform exponential stability of (2.14).

As announced in the Introduction, we will apply the perturbation theory of semigroup to discuss the uniform exponential stability of (2.10) or (2.14). So we first study the uniform exponential stability of the system

$$\begin{cases} \dot{y}_{1,j+\frac{1}{2}}(t) = \beta_2 \delta_s y_{2,j+\frac{1}{2}}(t), & \dot{y}_{2,j+\frac{1}{2}}(t) = \beta_1 \delta_s y_{1,j+\frac{1}{2}}(t), \\ \dot{y}_{3,j+\frac{1}{2}}(t) = \beta_4 \delta_s y_{4,j+\frac{1}{2}}(t), & \dot{y}_{4,j+\frac{1}{2}}(t) = \beta_3 \delta_s y_{3,j+\frac{1}{2}}(t), \\ y_{1,N+1}(t) = -\alpha_1 y_{2,N+1}(t), & y_{3,N+1}(t) = -\alpha_2 y_{4,N+1}(t), \\ y_{2,0}(t) = y_{4,0}(t) = 0, & j = 0, 1, \cdots, N. \end{cases} \tag{2.16}$$

we also represent (2.16) into vectorial form, let

$$\widetilde{\Psi}_h = \begin{pmatrix} 0 & \beta_2 B_h^\top & 0 & 0 \\ -\beta_1 B_h & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 B_h^\top \\ 0 & 0 & -\beta_3 B_h & 0 \end{pmatrix}.$$

Then we can write (2.16) as

$$\dot{Y}_h(t) = \widetilde{\mathcal{A}}_h Y_h(t), \quad Y_h(0) \in \mathbb{X}_h, \tag{2.17}$$

where $\widetilde{\mathcal{A}}_h = \Phi_h^{-1}(\widetilde{\Psi}_h + \Omega_h)$. Set $M_1 = \max\{1/\sqrt{\beta_1\beta_2},\ 1/\sqrt{\beta_3\beta_4}\}$ and the uniform exponential stability of (2.16) or (2.17) is presented as follows.

Theorem 2.2 For any initial value $Y_h(0) \in \mathbb{X}_h$, the energy $E_h(t)$ of the solution to (2.16) decays uniformly exponentially with respect to the step size, i.e., there exist two positive constants M and η independent of h such that

$$E_h(t) \le M e^{-\eta t} E_h(0), \text{ with } M = \frac{1+\varepsilon M_1}{1-\varepsilon M_1}, \ \eta = \frac{\varepsilon}{1+\varepsilon M_1}, \tag{2.18}$$

for $0 < \varepsilon < 1/M_1$.

Proof. Firstly, we introduce auxiliary functional

$$\varphi_h(t) = h \sum_{j=0}^{N} \left[s_{j+\frac{1}{2}} y_{1,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) + s_{j+\frac{1}{2}} y_{3,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t) \right]$$

and Lyapunov functional $G_h(t) = E_h(t) + \varepsilon \varphi_h(t)$. It is easy to see that $|\varphi_h(t)| \le M_1 E_h(t)$ and hence one has

$$(1 - \varepsilon M_1) E_h(t) \le G_h(t) \le (1 + \varepsilon M_1) E_h(t), \tag{2.19}$$

in which the condition $0 < \varepsilon < 1/M_1$ ensures that the Lyapunov functional $G_h(t)$ is positive definite.

Secondly, differentiating $\varphi_h(t)$ with respect to t and applying (2.16), we get

$$\begin{aligned} \dot{\varphi}_h(t) = {} & h \sum_{j=0}^{N} \left[\beta_1 s_{j+\frac{1}{2}} \delta_s y_{1,j+\frac{1}{2}}(t) y_{1,j+\frac{1}{2}}(t) + \beta_2 s_{j+\frac{1}{2}} \delta_s y_{2,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) \right] \\ & + h \sum_{j=0}^{N} \left[\beta_3 s_{j+\frac{1}{2}} \delta_s y_{3,j+\frac{1}{2}}(t) y_{3,j+\frac{1}{2}}(t) + \beta_4 s_{j+\frac{1}{2}} \delta_s y_{4,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t) \right]. \end{aligned} \tag{2.20}$$

Applying Lemma 2.4 to each term of the right-hand side of (2.20), we have

$$\begin{aligned} \dot{\varphi}_h(t) = {} & -E_h(t) - \frac{h^3}{8} \sum_{l=1}^{4} \sum_{j=0}^{N} \beta_l |\delta_s y_{l,j+\frac{1}{2}}(t)|^2 \\ & + \frac{1}{2} \left(\beta_2 + \beta_1 \alpha_1^2 \right) |y_{2,N+1}(t)|^2 + \frac{1}{2} \left(\beta_4 + \beta_3 \alpha_2^2 \right) |y_{4,N+1}(t)|^2. \end{aligned} \tag{2.21}$$

At last, differentiating the Lyapunov functional $G_h(t)$ and applying Lemma 2.3, we have

$$\dot{G}_h(t) = \dot{E}_h(t) + \varepsilon\dot{\varphi}_h(t) \le -\eta G_h(t),$$

since $\varepsilon \le 2\alpha_1\beta_1\beta_2/(\beta_1\alpha_1^2 + \beta_2) \le \sqrt{\beta_1\beta_2}$ and $\varepsilon \le 2\alpha_2\beta_3\beta_4/(\beta_4 + \alpha_2^2\beta_3) \le \sqrt{\beta_3\beta_4}$. Thus, applying the comparison principle (see [22] or [31, Section 2.3]) and (2.19), we obtain (2.18). $\square$

Furthermore, we define the disturbance operators $\mathcal{D}_h$ as

$$\mathcal{D}_h Y_h(t) = (-\beta_4 A_h^\top Y_{4h}(t), 0, 0, \beta_1 A_h Y_{1h}(t))^\top.$$

Then we have the following theorem.

Theorem 2.3 If ε is the same as in the last theorem and $2\sqrt{M\beta_1\beta_4} < \eta$ holds, then the system (2.10) or (2.14) is uniformly exponentially stable in the sense of Theorem 2.2.

Proof. Denote the C_0-semigroups generated by the operators $\mathcal{A}_h$ and $\widetilde{\mathcal{A}}_h$ as $T_h(t)$ and $T_{0h}(t)$, respectively. We should know the norms of the operators $\mathcal{D}_h$ to analyze the exponential stability of the semigroups $T_h(t)$ by the theory of bounded perturbations of C_0-semigroups [6, Theorem 10.3.1]. In fact, by the definitions of the operator $\mathcal{D}_h$, it is easy to see that for any $Z_h \in \mathbb{X}_h$,

$$\|\mathcal{D}_h Z_h\|_{\mathbb{X}_h}^2 = h\sum_{j=0}^{N}\left[\beta_1|\beta_4 z_{4,j+\frac{1}{2}}(t)|^2 + \beta_4|\beta_1 z_{1,j+\frac{1}{2}}(t)|^2\right].$$

It is not difficult to verify that $\|\mathcal{D}_h\|_{\mathbb{X}_h}^2 = \beta_1\beta_4$ holds. From Theorem 2.2, we know $T_{0h}(t)$ are uniformly exponentially stable since $2E_h(Y;t) = \|T_h(t)Y_h(0)\|_H^2$ for any $t \ge 0$. Whereas (2.18) implies that

$$\|T_{0h}(t)Y_h(0)\|_{\mathbb{X}_h}^2 = 2E_h(t) \le 2Me^{-\eta t}E_h(0) = Me^{-\eta t}\|Y_h(0)\|_{\mathbb{X}_h}^2,\ \forall\ Y_h(0) \in \mathbb{X}_h,$$

which is equivalent to $\|T_{0h}(t)\|_{\mathbb{X}_h} \le \sqrt{M}e^{-\eta t/2}$. Using [6, Theorem 10.3.1], we obtain

$$\|T_h(t)\|_{\mathbb{X}_h} \le \sqrt{M}e^{(-\eta/2+\sqrt{M}\|\mathcal{D}_h\|_{\mathbb{X}_h})t}.$$

The condition $2\sqrt{M\beta_1\beta_4} < \eta$ ensures $-\eta/2+\sqrt{M}\|\mathcal{D}_h\|_{\mathbb{X}_h} < 0$ which implies that the semigroups $T_h(t)$ are uniformly exponentially stable. $\square$

Remark 2.1 One may wonder whether or not there is ε such that all the conditions given in Theorem 2.2 and Theorem 2.3 are satisfied. In fact, let $\beta_1\beta_2 = \beta_3\beta_4 = 1$ hold, then one should choose β_1 and β_4 to satisfy

$$\beta_1\beta_4 < \frac{\varepsilon^2(1-\varepsilon)}{4(1+\varepsilon)^3}, \text{ with } 0 < \varepsilon < 1,$$

which ensure that $0 < \varepsilon < 1/M_1$ and $2\sqrt{M\beta_1\beta_4} < \eta$ hold. This manifests that $\beta_l (l = 1, 2, 3, 4)$ is not always suitable for the conditions given in Theorem 2.2 and Theorem 2.3, this is a restriction of this method.

Remark 2.2 Theorem 9.1.3 of [6], which is used to prove Theorem 2.1, is very effective to study the exponential stability of port-Hamiltonian system on infinite-dimensional space. To the best of our knowledge, there is no counterpart for uniform exponential stability of a family of port-Hamiltonian systems on finite-dimensional spaces. From the proof of Theorem 9.1.3 of [6](see also [3]), we know that the exponential stability highly relies on a kind of final state observability of a dissipative boundary control system. Thus, how to generalize this result to testify uniform final state observability of a family of dissipative boundary control systems is an open problem.

3. Related results of the beam (1.4)

In this section, we study the uniform exponential stability of Timoshenko beam (1.4) with interior damping. Two kinds of methods applied in this section are completely different in contrast with the methods of last section.

3.1. Exponential stability of continuous beam (1.4)

Using the auxiliary functions $y_l(s,t)(l=1,2,3,4)$ of subsection 2.1, we can transform (1.4) into the following system

$$\begin{cases} \dot{y}_1(s,t)=\beta_2 y_2'(s,t)-\beta_4 y_4(s,t), \\ \dot{y}_2(s,t)=\beta_1 y_1'(s,t)-\beta_2 y_2(s,t), \\ \dot{y}_3(s,t)=\beta_4 y_4'(s,t), \\ \dot{y}_4(s,t)=\beta_3 y_3'(s,t)+\beta_1 y_1(s,t)-\beta_4 y_4(s,t), \\ y_1(1,t)=y_2(0,t)=y_3(1,t)=y_4(0,t)=0. \end{cases} \tag{3.1}$$

The first four equations of (3.1) can be rewritten as the form

$$\dot{Y}(s,t)=P_1[\mathcal{H}Y(s,t)]'+[P_0+P]\mathcal{H}Y(s,t), \tag{3.2}$$

in which $\mathcal{H}$, P_0, and P_1 are defined in subsection 2.1 and $P=\mathrm{diag}(0,-1,0,-1)$.

Now, we consider the operator

$$A_0Y(s)=P_1(\mathcal{H}Y(s))'+P\mathcal{H}Y(s),\ \forall\ Y(s)\in D(A_0) \tag{3.3}$$

on the state space $\mathbb{X}$ and the domain

$$D(A_0)=\left\{Y\in\mathbb{X}:\mathcal{H}Y\in H^1([0,1];\mathbb{C}^4),\ W_B^0\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}=0\right\}, \tag{3.4}$$

in which $W_B^0\in\mathbb{C}^{4\times 8}$ is derived by setting α_1 and α_2 to be zero in W_B.

Let the operator B be defined by $BY(s)=P_0\mathcal{H}Y(s)$, for all $Y(s)\in\mathbb{X}$, then the abstract differential equation

$$Y(s,t)=(A_0+B)Y(s,t),\ Y(s,0)\in\mathbb{X} \tag{3.5}$$

is equivalent to (3.1). The operators A_0, B, and A_0+B satisfy the following properties.

Property 3.1 For any $Y\in D(A_0)$, we have

$$\mathrm{Re}\langle A_0Y,Y\rangle=-\int_0^1\beta_2^2|y_2(s,t)|^2+\beta_4^2|y_4(s,t)|^2ds. \tag{3.6}$$

Proof. In fact, for $Y\in D(A_0)$, it follows from the definition of A_0 and the inner product $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ given in last subsection that

$$\begin{aligned}\langle A_0Y,Y\rangle_{\mathbb{X}}=&\int_0^1\beta_1\beta_2y_2'(s,t)\overline{y}_1(s,t)+\beta_1\beta_2y_1'(s,t)\overline{y}_2(s,t)-\beta_2^2|y_2(s,t)|^2ds\\ &+\int_0^1\beta_3\beta_4y_4'(s,t)\overline{y}_3(s,t)+\beta_3\beta_4y_3'(s,t)\overline{y}_4(s,t)-\beta_4^2|y_4(s,t)|^2ds\end{aligned}$$

and

$$\begin{aligned}\langle Y,A_0Y\rangle_{\mathbb{X}}=&\int_0^1\beta_1\beta_2\overline{y'}_2(s,t)y_1(s,t)+\beta_1\beta_2\overline{y'}_1(s,t)y_2(s,t)-\beta_2^2|y_2(s,t)|^2ds\\ &+\int_0^1\beta_3\beta_4\overline{y'}_4(s,t)y_3(s,t)+\beta_3\beta_4\overline{y'}_3(s,t)y_4(s,t)-\beta_2^2|y_2(s,t)|^2ds.\end{aligned}$$

Adding two identities above together, we derive

$$\begin{aligned}\mathrm{Re}\langle A_0Y,Y\rangle_{\mathbb{X}} =&\frac{1}{2}\int_0^1 \beta_1\beta_2[y_2(s,t)\overline{y}_1(s,t)+y_1(s,t)\overline{y}_2(s,t)]'ds-\int_0^1 \beta_2^2|y_2(s,t)|^2+\beta_4^2|y_4(s,t)|^2ds\\&+\frac{1}{2}\int_0^1 \beta_3\beta_4[y_4(s,t)\overline{y}_3(s,t)+y_3(s,t)\overline{y}_4(s,t)]'ds,\end{aligned}$$

which yields (3.6) by $W_B^0\,[f_\partial^*,e_\partial^*]^*=0 \Leftrightarrow y_1(1,t)=y_2(0,t)=y_3(1,t)=y_4(0,t)=0$. □

Property 3.2 The operator B is skew-adjoint and hence $\mathrm{Re}\langle A_0Y,Y\rangle_{\mathbb{X}} = \mathrm{Re}\langle (A_0+B)Y,Y\rangle_{\mathbb{X}} \le 0$.

Proof. We only prove the first statement since $\mathrm{Re}\langle BY,Y\rangle_{\mathbb{X}}=0$ if operator B is skew-adjoint. Whereas, it is easy to see that

$$\begin{aligned}&\langle BY,Z\rangle_{\mathbb{X}}+\langle Y,BZ\rangle_{\mathbb{X}}\\=&\beta_1\beta_4\int_0^1 -y_4(s,t)\overline{y}_1(s,t)+y_1(s,t)\overline{y}_4(s,t)ds+\beta_1\beta_4\int_0^1 -y_1(s,t)\overline{y}_4(s,t)+y_4(s,t)\overline{y}_1(s,t)ds\\=&0,\end{aligned}$$

which shows that $B^*=-B$. □

Properties 3.1 and 3.2 imply that the operators A_0 and A_0+B are both dissipative and they further imply that the point spectral sets $\sigma_p(A_0)$ and $\sigma_p(A_0+B)$ of A_0 and A_0+B are contained in the closed left half-plane of $\mathbb{C}$. Moreover, we have more stronger results.

Property 3.3 The point spectral sets $\sigma_p(A_0)$ and $\sigma_p(A_0+B)$ of A_0 and A_0+B are contained in the open left half-plane of $\mathbb{C}$.

Proof. We only prove $\sigma_p(A_0)$ is contained in the open left half-plane of $\mathbb{C}$ since another proof is similar. To this end, we show now by a contradiction argument. If there exist $\beta\in\mathbb{R}$ and nonzero $Y\in\mathbb{X}$ such that $A_0Y=i\beta Y$, then it follows from (3.3) and (3.4) that

$$\begin{cases} i\beta y_1(s)=\beta_2y_2'(s), & i\beta y_2(s)=\beta_1y_1'(s)-\beta_2y_2(s),\\ i\beta y_3(s)=\beta_4y_4'(s), & i\beta y_4(s)=\beta_3y_3'(s)-\beta_4y_4(s),\\ y_1(1)=y_2(0)=0, & y_3(1)=y_4(0)=0.\end{cases}\tag{3.7}$$

It follows from $\mathrm{Re}\langle A_0Y,Y\rangle_{\mathbb{X}}=\mathrm{Re}\langle i\beta Y,Y\rangle_{\mathbb{X}}=0$ and Property 3.1 that $y_2(s)=y_4(s)=0$. Thus, (3.7) is reduced to

$$\begin{cases} i\beta y_1(s)=y_1'(s)=i\beta y_3(s)=y_3'(s)=0,\\ y_1(1)=y_2(0)=y_3(1)=y_4(0)=0.\end{cases}$$

Poincaré inequality, $y_1(1)=y_3(1)=0$ and $y_1'(s)=y_3'(s)=0$ imply that $y_1(s)=y_3(s)=0$. This leads to a contradiction and thus $\sigma_p(A_0)\cap i\mathbb{R}=\emptyset$ holds. □

Property 3.4 Let $f(s)=\beta_1^{-1}\beta_2^{-1}(s^2+\beta_2s)$ and $g(s)=\beta_3^{-1}\beta_4^{-1}(s^2+\beta_4s)$ be polynomials and $\mu_n=\pi/2+n\pi$ $(n\in\mathbb{Z})$, then the point spectral set $\sigma_p(A_0)$ is given by

$$\sigma_p(A_0)=\{\lambda: f(\lambda)=-\mu_n^2,\ n\in\mathbb{Z}\}\cup\{\lambda: g(\lambda)=-\mu_n^2,\ n\in\mathbb{Z}\},\tag{3.8}$$

and the corresponding eigenvectors form an orthonormal basis.

Proof. Assume $Y \in D(A_0)$ such that $A_0 Y = \lambda Y$, then it follows from (3.3) and (3.4) that

$$\begin{cases} \lambda y_1(s) = \beta_2 y_2'(s), & \lambda y_2(s) = \beta_1 y_1'(s) - \beta_2 y_2(s), \\ \lambda y_3(s) = \beta_4 y_4'(s), & \lambda y_4(s) = \beta_3 y_3'(s) - \beta_4 y_4(s), \\ y_1(1) = y_2(0) = 0, & y_3(1) = y_4(0) = 0. \end{cases}$$

The above equations are then equivalent to

$$\begin{cases} y_2''(s) = f(\lambda) y_2(s), & y_4''(s) = g(\lambda) y_4(s), \\ \beta_2 y_2'(s) = \lambda y_1(s), & \beta_4 y_4'(s) = \lambda y_3(s), \\ y_1(1) = y_2(0) = 0, & y_3(1) = y_4(0) = 0. \end{cases}$$

$y_2(s)$ and $y_4(s)$ can be easily solved by $y_2(0) = 0$ and $y_4(0) = 0$:

$$y_2(s) = C_1 \sinh(\sqrt{f(\lambda)} s), \quad y_4(s) = C_2 \sinh(\sqrt{g(\lambda)} s),$$

where C_1 and C_2 are the parameters determined later. Moreover, $y_1(1) = 0$ and $y_3(1) = 0$ implies $\sqrt{f(\lambda)} = i\mu_n$ and $\sqrt{g(\lambda)} = i\mu_n$ $(n \in \mathbb{Z})$. Thus, this means that (3.8) holds. More precisely, we obtain the point spectral set $\sigma_p(A_0)$ consisting of two branches $\sigma_p^1(A_0)$ and $\sigma_p^2(A_0)$, which are respectively given by

$$\sigma_p^1(A_0) = \left\{ \lambda_n^{\pm} = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1\beta_2\mu_n^2}}{2} \right\}, \ \sigma_p^2(A_0) = \left\{ \gamma_n^{\pm} = \frac{-\beta_4 \pm \sqrt{\beta_4^2 - 4\beta_3\beta_4\mu_n^2}}{2} \right\}, \ n \in \mathbb{Z}.$$

The eigenvectors $Y_n^{\pm}$ and $Z_n^{\pm}$ corresponding to eigenvalues $\lambda_n^{\pm}$ and $\gamma_n^{\pm}$ are

$$Y_n^{\pm} = C_{1n} \left[-\frac{1}{\lambda_n^{\pm}} \cos(\mu_n s), \sin(\mu_n s), 0, 0 \right] \ \text{ and } \ Z_n^{\pm} = C_{2n} \left[0, 0, -\frac{1}{\gamma_n^{\pm}} \cos(\mu_n s), \sin(\mu_n s) \right],$$

respectively. The parameters C_{1n} and C_{2n} are chosen such that $\|Y_n^{\pm}\|_{\mathbb{X}} = 1$ and $\|Z_n^{\pm}\|_{\mathbb{X}} = 1$. It follows from basic theory of function that $\{Y_n^{\pm} : n \in \mathbb{Z}\} \cup \{Z_n^{\pm} : n \in \mathbb{Z}\}$ forms orthonormal basis of $\mathbb{X}$. □

Under these preparations, we can prove an exponential stability result on the operator A_0.

Theorem 3.1 The operator A_0 generates a contractive semigroup which is exponentially stable.

Proof. We first show that A_0 generates a contractive semigroup. By Property 3.1, we know that A_0 is dissipative. Thus, it is sufficient to to show that $\mathrm{Ran}(I - A_0) = X$ by Lumer-Phillips Theorem. That is to say, for each $Z \in X$ we have to find $Y \in D(A_0)$ such that $(I - A_0)Y = Z$, which is equivalent to

$$\begin{cases} y_1(s) - \beta_2 y_2'(s) = z_1(s), & (1+\beta_2) y_2(s) - \beta_1 y_1'(s) = z_2(s), \\ y_3(s) - \beta_4 y_4'(s) = z_3(s), & (1+\beta_4) y_4(s) - \beta_3 y_3'(s) = z_4(s), \\ y_1(1) = y_2(0) = 0, & y_3(1) = y_4(0) = 0. \end{cases}$$

By the same method of [6, Example 6.1.10], we can obtain unique Y given by

$$\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} \delta_1 \sinh(\gamma_1 s) \\ (2\beta_2\gamma_1)^{-1} \delta_1 \sinh(\gamma_1 s) \end{pmatrix} + \int_0^s \Phi_1(s-\tau)(z_1(s-\tau), \ z_2(s-\tau))^{\top} d\tau,$$
$$\begin{pmatrix} y_3(s) \\ y_4(s) \end{pmatrix} = \begin{pmatrix} \delta_2 \sinh(\gamma_2 s) \\ (2\beta_4\gamma_2)^{-1} \delta_2 \sinh(\gamma_2 s) \end{pmatrix} + \int_0^s \Phi_2(s-\tau)(z_3(s-\tau), \ z_4(s-\tau))^{\top} d\tau,$$

in which $\gamma_1 = \sqrt{\beta_1^{-1}\beta_2^{-1}(1+\beta_2)}$, $\gamma_2 = \sqrt{\beta_3^{-1}\beta_4^{-1}(1+\beta_4)}$,

$$\delta_1 = -\frac{1}{\sinh(\gamma_1)} \int_0^1 \sinh(\gamma_1(s-\tau)) z_1(s-\tau) + \beta_2\gamma_1 \sinh(\gamma_1(s-\tau)) z_2(s-\tau) d\tau,$$

$$\delta_2 = -\frac{1}{\sinh(\gamma_2)}\int_0^1 \frac{1}{2\beta_4\gamma_2}\cosh(\gamma_2(s-\tau))z_3(s-\tau)+\cosh(\gamma_2(s-\tau))z_4(s-\tau)d\tau,$$

$$\Phi_1(s) = \begin{pmatrix} \sinh(\gamma_1 s) & \beta_2\gamma_1\sinh(\gamma_1 s) \\ (2\beta_2\gamma_1)^{-1}\cosh(\gamma_1 s) & \cosh(\gamma_1 s)\end{pmatrix},$$

$$\Phi_2(s) = \begin{pmatrix} \sinh(\gamma_2 s) & \beta_4\gamma_2\sinh(\gamma_2 s) \\ (2\beta_4\gamma_2)^{-1}\cosh(\gamma_2 s) & \cosh(\gamma_2 s)\end{pmatrix}.$$

On the one hand, this means that $(I-A_0)$ is bijective and hence A_0 generates a contractive semigroup. On the other hand, it follows from [32, Section 22.2] that $(I-A_0)^{-1}$ is a compact operator. Therefore, the spectral set $\sigma(A_0)$ and the point spectral set $\sigma_p(A_0)$ coincide and Property 3.4 further implies spectrum determined growth assumption holds. The exponential stability of the semi-group generated by A_0 is finally derived since Property 3.4 manifests that the spectral bound of A_0 is negative. □

Applying this theorem and [13, Theorem 5.1], we can prove the exponential stability of the semi-group generated by the operator A_0+B. To this end, we introduce the definition of B is A-compact since it is involved in [13, Theorem 5.1].

Definition 3.1 Let A and B defined on Banach space X be two linear operators such that $D(A)\subset D(B)$, we shall say that B is A-compact, if for any sequence $(x_n)_n\subset D(A)$ with both $(x_n)_n$ and $(Ax_n)_n$ bounded, $(Bx_n)_n$ contains a convergent subsequence.

Theorem 3.2 The operator A_0+B generates an exponentially stable semigroup.

Proof. Theorem 5.1 of [13] asserts that the semigroup generated by A_0+B is exponential stable if and only if $\sigma_p(A_0)$ lies in the open left-half plane if B is A_0-compact. But, we have proven that $\sigma_p(A_0)$ is contained in the open left-half plane in Property 3.3. It is sufficient to verify that B is A_0-compact. Thus, let $(Y_n(s,t))_n\subset D(A)$ with both $(Y_n(s,t))_n$ and $(A_0Y_n(s,t))_n$ bounded, we have $(y_{ln}(s,t))_n$ and $(y'_{ln}(s,t))_n$ are uniformly bounded for $l=1,2,3,4$ by the definition of the operator A_0. Combining Theorem 2 of [32, Section 22.1], which is Rellich's standard of precompact set in $L^2(\Omega)$ with Ω being the open set of $\mathbb{R}^n$, and $BY=(-\beta_4y_4(s,t),0,0,\beta_1y_1(s,t))$, we know that B is A_0-compact. □

3.2. Uniform exponential stability of (3.1)

To study the uniform exponential stability of (3.1), we discretize it by the average central-difference method of section 2.2. It is easy to know that the semi-discretized system of (3.1) is

$$\begin{cases} \dot w_{1,j+\frac12}(t) = \beta_2\delta_x w_{2,j+\frac12}(t) - \beta_4 w_{4,j+\frac12}(t), & t>0, \\ \dot w_{2,j+\frac12}(t) = \beta_1\delta_x w_{1,j+\frac12}(t) - \beta_2 w_{2,j+\frac12}(t), & j=0,1,\cdots,N, \\ \dot w_{3,j+\frac12}(t) = \beta_4\delta_x w_{4,j+\frac12}(t), & \\ \dot w_{4,j+\frac12}(t) = \beta_3\delta_x w_{3,j+\frac12}(t) + \beta_1 w_{1,j+\frac12}(t) - \beta_4 w_{4,j+\frac12}(t), & \\ w_{1,N+1}(t) = w_{2,0}(t) = w_{3,N+1}(t) = w_{4,0}(t) = 0. & \end{cases} \tag{3.9}$$

Let $W_h(t) = (W_{1h}, W_{2h}, W_{3h}, W_{4h})^\top$ be unknown variables consisting of

$$W_{1h}(t) = (w_{1,0}(t),\ \cdots,\ w_{1,N}(t)),\ W_{2h}(t) = (w_{2,1}(t),\ \cdots,\ w_{2,N+1}(t)),$$

$$W_{3h}(t) = (w_{3,0}(t),\ \cdots,\ w_{3,N}(t)),\ W_{4h}(t) = (w_{4,1}(t),\ \cdots,\ w_{4,N+1}(t)).$$

A_h, B_h, and $\mathcal{H}_h$ are given in the last section, we therefore define the matrix

$$G_h = \begin{pmatrix} 0 & B_h^\top & 0 & -A_h^\top \\ -B_h & -A_h^\top & 0 & 0 \\ 0 & 0 & 0 & B_h^\top \\ A_h & 0 & -B_h & -A_h^\top \end{pmatrix},$$

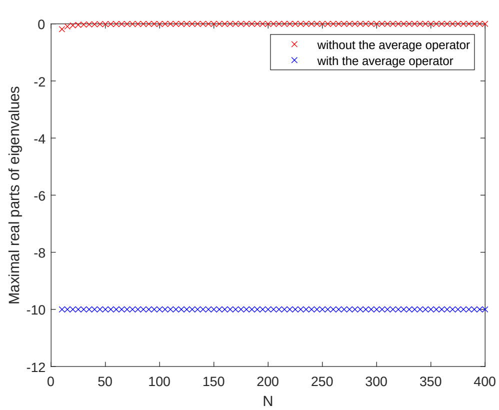
Using these matrices, we rewrite (3.9) into a vector form

$$\Phi_h \dot{W}_h(t) = G_h \mathcal{H}_h W_h(t). \tag{3.10}$$

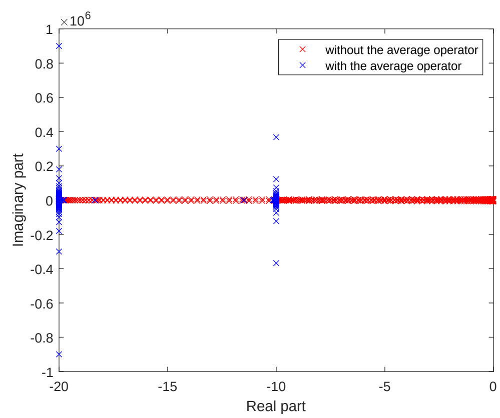
The state spaces of (3.9) is still denoted by $\mathbb{X}_h$ which is the linear space $\mathbb{C}^{4(N+1)}$ equipped with the inner product $\langle W_h, V_h \rangle_{\mathbb{X}_h} = h\langle \Phi_h W_h, \mathcal{H}_h \Phi_h V_h \rangle$ for $W_h, V_h \in \mathbb{X}_h$, in which $\langle \cdot, \cdot \rangle$ is natural inner product of $\mathbb{C}^{4(N+1)}$ or $\mathbb{C}^{N+1}$. The classical finite difference scheme of (3.1) is easily obtained from (3.10),

$$\dot{W}_h(t) = G_h \mathcal{H}_h W_h(t). \tag{3.11}$$

Similarly as in subsection 2.2, we also plot two figures to explain the effective of our numerical approximating scheme (3.10). Figure 3 depicts the maximal real parts of the eigenvalues of our discrete scheme (3.10) and the classical semi-discrete scheme (3.11) for $N = 10 : 5 : 400$. Figure 4 depicts the distributions of the eigenvalues of (3.10) and (3.11) in which $N = 100$. In both figures, we take $\beta_1 = 10,\ \beta_2 = 20,\ \beta_3 = 30,\ \beta_4 = 40$. We can draw the same conclusion with those of subsection 2.2. That is to say (3.10) is uniformly exponentially stable and (3.11) does not possess this property.



(c) Fig.3. Maximal real parts of eigenvalues



(d) Fig.4. Eigenvalue distributions

To prove (3.10) is uniformly exponentially stable, we firstly show that $\widetilde{A}_h := \Phi_h^{-1} G_h \mathcal{H}_h$ is dissipative for every step size h.

Lemma 3.1 The operator $\widetilde{A}_h := \Phi_h^{-1} G_h \mathcal{H}_h$ is dissipative on the space $\mathbb{X}_h$, i.e., for $h \in (0,1)$ and $Y_h \in \mathbb{X}_h$,

$$\mathrm{Re}\langle \widetilde{A}_h Y_h, Y_h \rangle_{\mathbb{X}_h} = -\beta_2^2 h \langle A_h^\top Y_{2h}, A_h^\top Y_{2h} \rangle - \beta_4^2 h \langle A_h^\top Y_{4h}, A_h^\top Y_{4h} \rangle \le 0. \tag{3.12}$$

Proof. Here and hereafter, we assume $Y_h = (Y_{1h}, Y_{2h}, Y_{3h},\ Y_{4h})^\top \in \mathbb{X}_h$ with

$$\begin{aligned} Y_{1h}(t) = (y_{1,0}(t), \cdots, y_{1,N}(t)),\ Y_{2h}(t) = (y_{2,1}(t), \cdots, y_{2,N+1}(t)), \\ Y_{3h}(t) = (y_{3,0}(t), \cdots, y_{3,N}(t)),\ Y_{4h}(t) = (y_{4,1}(t), \cdots, y_{4,N+1}(t)), \end{aligned}$$

but we need to assign some imaginary components such as $y_{1,N+1}(t) = y_{2,0}(t) = y_{3,N+1}(t) = y_{4,0}(t) = 0$. By the definition of the operator $\widetilde{A}_h$ and the definition of the inner product $\langle \cdot, \cdot \rangle_{\mathbb{X}_h}$, we have

$$\begin{aligned} &\langle \widetilde{A}_h Y_h, Y_h \rangle_{\mathbb{X}_h} = h \langle G_h \mathcal{H}_h Y_h, \mathcal{H}_h \Phi_h Y_h \rangle \\ =&\beta_1\beta_2 h \langle B_h^\top Y_{2h}, A_h Y_{1h} \rangle - \beta_1\beta_2 h \langle B_h Y_{1h}, A_h^\top Y_{2h} \rangle - \beta_2^2 h \langle A_h^\top Y_{2h}, A_h^\top Y_{2h} \rangle \\ &+ \beta_3\beta_4 h \langle B_h^\top Y_{4h}, A_h Y_{3h} \rangle - \beta_3\beta_4 h \langle B_h Y_{3h}, A_h^\top Y_{4h} \rangle - \beta_4^2 h \langle A_h^\top Y_{4h}, A_h^\top Y_{4h} \rangle. \end{aligned} \tag{3.13}$$

and

$$
\begin{aligned}
&\langle Y_h, \widetilde{A}_h Y_h\rangle_{\mathbb{X}_h} = h\langle \Phi_h Y_h, \mathcal{H}_h G_h \mathcal{H}_h Y_h\rangle \\
=&\beta_1\beta_2 h\langle A_h Y_{1h}, B_h^\top Y_{2h}\rangle - \beta_1\beta_2 h\langle A_h^\top Y_{2h}, B_h Y_{1h}\rangle - \beta_2^2 h\langle A_h^\top Y_{2h}, A_h^\top Y_{2h}\rangle \\
&+ \beta_3\beta_4 h\langle A_h Y_{3h}, B_h^\top Y_{4h}\rangle - \beta_3\beta_4 h\langle A_h^\top Y_{4h}, B_h Y_{3h}\rangle - \beta_4^2 h\langle A_h^\top Y_{4h}, A_h^\top Y_{4h}\rangle.
\end{aligned}
\tag{3.14}
$$

By some simple calculations, we know that

$$
\begin{aligned}
&h\langle B_h^\top Y_{2h}, A_h Y_{1h}\rangle - h\langle A_h^\top Y_{2h}, B_h Y_{1h}\rangle \\
=&\sum_{j=0}^{N} \frac{(y_{2,j+1} - y_{2,j})(\overline{y}_{1,j+1} + \overline{y}_{1,j})}{2} + \sum_{j=0}^{N} \frac{(y_{1,j+1} - y_{1,j})(\overline{y}_{2,j+1} + \overline{y}_{2,j})}{2} \\
=&y_{1,N+1}\overline{y}_{2,N+1} - y_{1,0}\overline{y}_{2,0} = 0.
\end{aligned}
\tag{3.15}
$$

Similarly, we also have

$$
\left\{
\begin{array}{l}
-h\langle B_h Y_{1h}, A_h^\top Y_{2h}\rangle + h\langle A_h Y_{1h}, B_h^\top Y_{2h}\rangle = 0 \\
h\langle B_h^\top Y_{4h}, A_h Y_{3h}\rangle - h\langle A_h^\top Y_{4h}, B_h Y_{3h}\rangle = 0. \\
-h\langle B_h Y_{3h}, A_h^\top Y_{4h}\rangle + h\langle A_h Y_{3h}, B_h^\top Y_{4h}\rangle = 0
\end{array}
\right.
\tag{3.16}
$$

Thus, (3.13)-(3.16) implies that (3.12) is right since $\langle \widetilde{A}_h Y_h, Y_h\rangle_{\mathbb{X}_h} + \langle Y_h, \widetilde{A}_h Y_h\rangle_{\mathbb{X}_h} = 2\mathrm{Re}\langle \widetilde{A}_h Y_h, Y_h\rangle_{\mathbb{X}_h}$. □

From last lemma, we know that $\widetilde{A}_h$ generates a family of semigroups of contraction $T_h(t)$. In the sequel, we apply frequency standard of uniform exponential stability for a family of contractive semigroups, which is given in [15], [29] or [30], to study the uniform exponential stability of $T_h(t)$ for all $h \in (0, 1)$.

Theorem 3.3 Let $h^* > 0$ and $(S_h(t))$ be a family of semigroups of contraction on the Hilbert space $(\widetilde{\mathbb{X}}_h)$, and let $(\widetilde{A}_h)$ be the corresponding infinitesimal generators. The family $(S_h(t))$ is uniformly exponentially stable if and only if the two following conditions are satisfied:

(i) For all $h \in (0, h^*)$, $i\mathbb{R}$ is contained in the resolvent set $\rho(\widetilde{A}_h)$ of $(\widetilde{A}_h)$.

(ii) $\sup_{h\in(0,h^*),\beta\in\mathbb{R}} \|(i\beta I - \widetilde{A}_h)^{-1}\|_{L(\widetilde{\mathbb{X}}_h)} < \infty$.

We first show that (i) holds for the operator $\widetilde{A}_h$.

Lemma 3.2 For all $h \in (0, 1)$, $\{i\beta,\ \beta \in \mathbb{R}\} \subset \rho(\widetilde{A}_h)$.

Proof. Suppose this conclusion is false, i.e., $i\beta \in \sigma(\widetilde{A}_h)$, then there exists $0 \neq Y_h \in \mathbb{X}_h$ and $\beta \in \mathbb{R}$ such that $\widetilde{A}_h Y_h = i\beta Y_h$. On the one hand, it is easy to see that $\mathrm{Re}\langle \widetilde{A}_h Y_h, Y_h\rangle_{\mathbb{X}_h} = \mathrm{Re}\langle i\beta Y_h, Y_h\rangle_{\mathbb{X}_h} = 0$. By (3.12), we know that $\langle A_h^\top Y_{2h}, A_h^\top Y_{2h}\rangle = 0$ and $\langle A_h^\top Y_{4h}, A_h^\top Y_{4h}\rangle = 0$, which implies that $Y_{2h} = Y_{4h} = 0$, since B_h is invertible. On the other hand, $\widetilde{A}_h Y_h = i\beta Y_h$ is reduced to

$$
i\beta A_h Y_{1h} = 0,\ \ -\beta_1 B_h Y_{1h} = 0,\ \beta_1 A_h Y_{1h} - \beta_3 B_h Y_{3h} = 0,
$$

which yield that $Y_{1h} = Y_{3h} = 0$ since B_h are invertible. Thus $Y_h = 0$ is derived and this contradicts $Y_h \neq 0$. □

Now we can present the main result of this subsection.

Theorem 3.4 The family $(T_h(t))$ is uniformly exponentially stable, that is to say there exist two constants $M > 0$ and $\omega > 0$ (independent of $h \in (0, 1)$) such that

$$
\|T_h(t)\|_{L(\mathbb{X}_h)} \leq M e^{-\omega t},\ \forall t \geq 0.
\tag{3.17}
$$

Proof The proof is based on Theorem 3.3. Notice first that, for all $h \in (0,1)$, the family $(T_h(t))$ form a family of contraction semigroups (see Lemma 3.1). The fact that the family $(\widetilde{A}_h)$ satisfies condition (i) follows from Lemma 3.2. In order to show that the family $(\widetilde{A}_h)$ satisfies condition (ii) we use a contradiction argument. If the condition (ii) is false, then there exist $b_n \in \mathbb{R}$, $h_n \in (0,1)$, and $Z^n_{h_n} \in \mathbb{X}_{h_n}$ such that

$$\|(ib_nI_{h_n} - \widetilde{A}_{h_n})^{-1}Z^n_{h_n}\|_{\mathbb{X}_{h_n}} \geq n\nu_n h_n^{-1}\|Z^n_{h_n}\|_{\mathbb{X}_{h_n}},$$

in which $\nu_n = |ib_n + \max\{\beta_2, \beta_4\}|^2$, $n = 1, 2, \cdots$, and the norm $\|\cdot\|_{\mathbb{X}_{h_n}}$ is induced by the inner product of $\mathbb{X}_{h_n}$. Let $(ib_nI_{h_n} - \widetilde{A}_{h_n})^{-1}Z^n_{h_n} = F^n_{h_n}$ and we have

$$\|F^n_{h_n}\|_{\mathbb{X}_{h_n}} \geq n\nu_n h_n^{-1}\|(ib_nI_{h_n} - \widetilde{A}_{h_n})F^n_{h_n}\|_{\mathbb{X}_{h_n}}.$$

Setting $Y^n_{h_n} = F^n_{h_n}/\|F^n_{h_n}\|_{\mathbb{X}_{h_n}}$, we obtain $\|Y^n_{h_n}\|_{\mathbb{X}_{h_n}} = 1$ and

$$\|U^n_{h_n}\|_{\mathbb{X}_{h_n}} \leq n^{-1}\nu_n^{-1}h_n, \text{ with } U^n_{h_n} = (ib_nI_{h_n} - \widetilde{A}_{h_n})Y^n_{h_n}. \tag{3.18}$$

More precisely, for $U^n_{h_n} = (U^n_{1h_n}, U^n_{2h_n}, U^n_{3h_n}, U^n_{4h_n})$, we have

$$\begin{aligned}\|U^n_{h_n}\|^2_{\mathbb{X}_{h_n}} =& h_n\left[\beta_1\|A_{h_n}U^n_{1h_n}\|^2 + \beta_2\|A^\top_{h_n}U^n_{2h_n}\|^2 + \beta_3\|A_{h_n}U^n_{3h_n}\|^2 + \beta_4\|A^\top_{h_n}U^n_{4h_n}\|^2\right]\\ \leq& \nu_n^{-2}n^{-2}h_n^2.\end{aligned} \tag{3.19}$$

It follows from Cauchy-Schwartz inequality, (3.12), and (3.18) that

$$\begin{aligned}\mathrm{Re}\left\langle U^n_{h_n}, Y^n_{h_n}\right\rangle_{\mathbb{X}_{h_n}} =& -\mathrm{Re}\left\langle \widetilde{A}_{h_n}Y^n_{h_n}, Y^n_{h_n}\right\rangle_{\mathbb{X}_{h_n}}\\ =& h_n\left[\beta_2^2\|A^\top_{h_n}Y^n_{2h_n}\|^2 + \beta_4^2\|A^\top_{h_n}Y^n_{4h_n}\|^2\right] \leq \nu_n^{-1}n^{-1}h_n.\end{aligned} \tag{3.20}$$

This means that

$$h_n\|A^\top_{h_n}Y^n_{2h_n}\|^2 = O(\nu_n^{-1}n^{-1}h_n), \quad h_n\|A^\top_{h_n}Y^n_{4h_n}\|^2 = O(\nu_n^{-1}n^{-1}h_n), \tag{3.21}$$

in which the notation $f_n = O(n^{-1})$ denotes that there is a positive constant C (independent of n) such that $\|f_n\| \leq Cn^{-1}$ for $n \in \mathbb{Z}^+$. Now we consider the identity $U^n_{h_n} = (ib_nI_{h_n} - \widetilde{A}_{h_n})Y^n_{h_n}$, which can be expanded as

$$\begin{aligned}ib_nA_hY^n_{1h_n} - \beta_2B_h^\top Y^n_{2h_n} + \beta_4A_h^\top Y^n_{4h_n} =& A_hU^n_{1h_n}\\ \beta_1B_hY^n_{1h_n} + (ib_n + \beta_2)A_h^\top Y^n_{2h_n} =& A_h^\top U^n_{2h_n}\\ ib_nA_hY^n_{3h_n} - \beta_4B_h^\top Y^n_{4h_n} =& A_hU^n_{3h_n}\\ -\beta_1A_hY^n_{1h_n} - \beta_3B_hY^n_{3h_n} + (ib_n + \beta_4)A_h^\top Y^n_{4h_n} =& A_h^\top U^n_{4h_n}.\end{aligned}$$

More precisely, we have

$$ib_ny^n_{1,j+\frac{1}{2}} - \beta_2\delta_xy^n_{2,j+\frac{1}{2}} + \beta_4\delta_{\frac{1}{2}}y^n_{4,j+\frac{1}{2}} = u^n_{1,j+\frac{1}{2}}, \tag{3.22}$$

$$(ib_n + \beta_2)y^n_{2,j+\frac{1}{2}} - \beta_1\delta_xy^n_{1,j+\frac{1}{2}} = u^n_{2,j+\frac{1}{2}}, \tag{3.23}$$

$$ib_ny^n_{3,j+\frac{1}{2}} - \beta_4\delta_xy^n_{4,j+\frac{1}{2}} = u^n_{3,j+\frac{1}{2}}, \tag{3.24}$$

$$(ib_n + \beta_4)y^n_{4,j+\frac{1}{2}} - \beta_3\delta_xy^n_{3,j+\frac{1}{2}} - \beta_1y^n_{1,j+\frac{1}{2}} + = \delta_{\frac{1}{2}}u^n_{4,j+\frac{1}{2}}, \tag{3.25}$$

for $j = 0, 1, \cdots, N_n$ and $y^n_{1,N_n+1} = y^n_{2,0} = y^n_{3,N_n+1} = y^n_{4,0} = 0$. It follows from (3.23) that

$$\beta_1y^n_{1,N_n} = h_nu^n_{2,N_n+\frac{1}{2}} - h_n(ib_n + \beta_2)y^n_{2,N_n+\frac{1}{2}},$$

holds by setting $j=N_n$. (3.19) and (3.21) further imply that

$$|y^n_{1,N_n}|^2 \le \frac{2}{\beta_1^2}|ib_n+\beta_2|^2h_n^2|y^n_{2,N_n+\frac{1}{2}}|^2+\frac{2}{\beta_1^2}h_n^2|u^n_{2,N_n+\frac{1}{2}}|^2 \le C|ib_n+\max\{\beta_2,\beta_4\}|^2\nu_n^{-1}n^{-1}h_n$$

which can be expressed as

$$|y^n_{1,N_n}|^2 = O(n^{-1}h_n),$$

since $\nu_n=|ib_n+\max\{\beta_2,\beta_4\}|^2$. Similarly, by setting $j=N_n-1$ in (3.23), we get

$$\beta_1 y^n_{1,N_n-1}=\beta_1 y^n_{1,N_n}-h_n(ib_n+\beta_2)y^n_{2,N_n-\frac{1}{2}}+h_n\delta_{\frac{1}{2}}u^n_{2,N_n-\frac{1}{2}},$$

and

$$|y^n_{1,N_n-1}|^2 = O(n^{-1}h_n).$$

We can repeat this process to obtain

$$|y^n_{1,j}|^2 = O(n^{-1}h_n),\ \text{for all } j=0,1,\cdots,N_n,$$

which implies that

$$h_n\|A_hY^n_{1h_n}\|^2 = h_n\sum_{j=0}^{N_n}|y^n_{1,j+\frac{1}{2}}|^2 = O(n^{-1}), \tag{3.26}$$

since $h_n(N_n+1)=1$. Using (3.25) and the same method as above, we know that $Y^n_{3h_n}$ also satisfies

$$|y^n_{3,j}|^2 = O(n^{-1}h_n),\ \text{for all } j=0,1,\cdots,N_n,$$

and

$$h_n\|A_h^\top Y^n_{3h_n}\|^2 = h_n\sum_{j=0}^{N_n}|y^n_{3,j+\frac{1}{2}}|^2 = O(n^{-1}). \tag{3.27}$$

Finally, it follows from that (3.21), (3.26), and (3.27) that

$$\|Y^n_{h_n}\|^2_{\mathbb{X}_{h_n}} = h_n\left[\beta_1\|A_{h_n}Y^n_{1h_n}\|^2+\beta_2\|A^\top_{h_n}Y^n_{2h_n}\|^2+\beta_3\|A_{h_n}Y^n_{3h_n}\|^2+\beta_4\|A^\top_{h_n}Y^n_{4h_n}\|^2\right]=O(n^{-1}),$$

which leads to a contradiction since $\|Y^n_{h_n}\|_{\mathbb{X}_{h_n}}=1$ hold for all positive integers n. The proof of the Theorem 3.4 is complete. □

4. Concluding remarks

Timoshenko beam is a basic vibration model and plays important role in engineering. This paper is devoted to uniformly exponentially stable approximations for the Timoshenko beam with both interior damping and boundary damping. This means that we study it from the viewpoints of the numerical approximating and control theory. It is well-known that there are many discretization methods to discretize the spatial variables. It is not an easy job to pick one which preserves exponential stability among so many semi-discretization methods. Moreover, Timoshenko beam is described by two coupled wave equation and the crossed terms bring troubles in verifying the uniform exponential stability. To bypass these challenges, perturbation of exponentially stable C_0-semigroup is introduced and uniform exponential stability of both continuous and semi-discretization systems are then smoothly obtained. These results have potential applications in uniform controllability, the approximation of the control problem and state reconstruction etc. It is worthy to be investigated at length in the further research.

Acknowledgements

The authors would like to thank the anonymous reviewers for their very careful reading and valuable comments and suggestions to improve the manuscript.

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